On the divergences of inflationary superhorizon perturbations

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The "ΔN formalism"

On long wavelengths

$$ds^2 = dt^2 - a^2(t, \mathbf{x}) \mathsf{G}_{ij}(\mathbf{x}) dx^i dx^j$$

The exact Curvature Perturbation:

$$\zeta_i = \partial_i \ln a - \frac{H}{\dot\rho} \partial_i \rho$$

Evaluated on time slices defined by the energy density ($\partial_i \rho = 0$)

$$\zeta_i = \partial_i \ln a$$

Gives the "ΔN formalism":

$$\boxed{\ln a(\mathbf{x}) \equiv N(\phi_\star(\mathbf{x}))}$$

A spatial modulation of the initial field value $\phi_\star \longrightarrow \phi_\star(\mathbf{x})$ leads to a spatial modulation of the local expansion on time-slices of homogeneous energy density.

$$N(\phi_\star(\mathbf{x})) = N(\bar\phi_\star) + \underbrace{N_A \delta\phi_\star^A(\mathbf{x}) + \frac{1}{2} N_{AB} \delta\phi_\star^A(\mathbf{x}) \delta\phi_\star^B(\mathbf{x}) + \ldots}_{\zeta(\mathbf{x})}$$

IR divergences

Example: Single field (scale invariant) $\langle\phi_\star(\mathbf{x}_1)\phi_\star(\mathbf{x}_2)\rangle = G(|\mathbf{x}_1-\mathbf{x}_2|)$

$$\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle_{(1)} = (N')^2\, G(|\mathbf{x}_1-\mathbf{x}_2|) \sim -(N')^2\mathcal{P}_\phi \ln\left(\frac{|\mathbf{x}_1-\mathbf{x}_2|}{L}\right)$$

Difference doesn't depend on the IR cutoff:

$$\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle_{(1)} - \langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_3)\rangle_{(1)} \sim -(N')^2\mathcal{P}_\phi \ln\left|\frac{\mathbf{x}_1-\mathbf{x}_2}{\mathbf{x}_1-\mathbf{x}_3}\right|$$

At second order however:

$$\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle_{(2)} \sim -N'N'''\langle\phi^2\rangle\mathcal{P}_\phi \ln\left(\frac{|\mathbf{x}_1-\mathbf{x}_2|}{L}\right) + \frac{1}{2}(N'')^2\mathcal{P}^2\ln^2\left(\frac{|\mathbf{x}_1-\mathbf{x}_2|}{L}\right)$$

Two issues

- IR divergence of $G(|\mathbf{x}_1-\mathbf{x}_2|)$? (assumption of scale invariance)
- Suppose that there is a physical cutoff L. How does a prediction of the theory with cutoff L relate to observations up to scales $M < L$?

Stochastic Inflation

On super-hubble scales, the dynamics of the inflaton are adequately described by a stochastic probability distribution $P(\phi, N)$ which obeys $(N = \ln a)$

$$\frac{\partial P}{\partial N} = -\frac{1}{3\pi M_P^2}\frac{\partial^2}{\partial\phi^2}(VP) + \frac{M_P^2}{8\pi}\frac{\partial}{\partial\phi}\left(V^{-1}\frac{\partial V}{\partial\phi}P\right)$$

For $N \longrightarrow \infty$, P reaches a time independent asymptotic:

$$P(\phi) \sim V(\phi)^{-1}\exp\left(\frac{3M_P^4}{8V(\phi)}\right)$$

with

$$\langle\phi^n\rangle_{\rm IR} \propto \int_{\phi_{\rm min}}^{\phi_{\rm max}} d\phi\phi^n P(\phi)$$

The infrared ϕ correlators remain finite albeit large. However, this amount of non-Gaussianity will never be accessible for an observer within a given thermalized Hubble patch since the observer's horizon will never cross the hypersurface of eternal inflation:

$$V(\phi_{\rm EI}) \gtrsim \epsilon(\phi_{\rm EI})M_P^4$$

The value $\phi_{\rm EI}$ sets an upper bound for observationally interesting scales.

Thus, on very large scales:

- 1-point functions of the field and the curvature perturbation are regularized by stochastic inflation.
- There is a maximal scale associated with the surface of eternal inflation.

Below the scale of eternal inflation the concept of an average background evolution makes sense and one can use a perturbative approach to calculate n-point functions.

Renormalization

Suppose that there exists an IR cutoff L. Then, to one loop:

- $\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle = \left(N_A N^A + N_A N^{AB}_B \langle\phi^2\rangle^{(l)}_{(L)}\right) G_{12} + \frac{1}{2} N_{AB} N^{AB} G^2_{12}$

- $\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\rangle =$
$\left[N_{AB} N^A N^B + \left(\frac{1}{2} N^A_{ABC} N^B N^C + N^A_{AB} N^{BC} N_C\right) \langle\phi^2\rangle^{(l)}_{(L)}\right] (G_{12} G_{13} + \ldots) +$
$\frac{1}{2} N_{ABC} N^{AB} N^C \left((G_{12} + G_{13})\, G_{12} G_{13} + \ldots \right) + N_{AB} N^{BC} N^A_C \; G_{12} G_{13} G_{23}$

where

$$\langle\phi^2\rangle^{(l)}_{(L)} = \int_{1/L}^{1/l} \frac{dk}{k} \mathcal{P}_\phi$$

$$\langle\phi^A_\star(\mathbf{x}_i)\phi^B_\star(\mathbf{x}_j)\rangle \equiv \delta^{AB} G_{ij} = \delta^{AB} \int_{1/L}^{1/l} \frac{dk}{k} \mathcal{P}_\phi \frac{\sin k r_{ij}}{k r_{ij}}$$

How do these theoretical predictions relate to observations restricted to scales $M < L$?

For an observer located at some random position in the box of size L and restricted to measuring fluctuations up to scales $M < L$ (state translationally invariant)

$$G(r) \rightarrow \tilde{G}(r) \equiv G(r) - \langle \phi^2 \rangle_{(L)}^{(M)}$$

In terms of $\tilde{G}(r)$ and to one loop:

<u>2-Point Function</u>

$$\langle \zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle = \left(N_A N^A + \frac{1}{2}(N_A N^A)_B^B \langle \phi^2 \rangle_{(L)}^{(M)} + N_A N_B^{AB} \langle \phi^2 \rangle_{(M)}^{(l)} \right) \tilde{G}_{12} + \frac{1}{2} N_{AB} N^{AB} \tilde{G}_{12}^2 + \underbrace{N_A N^A \langle \phi^2 \rangle_{(L)}^{(M)}}_{\text{Constant}}$$

The coefficient of $\tilde{G}$ to one loop:

$$N_A N^A \Big|_{\bar{\phi}_L} + \frac{1}{2}(N_A N^A)_B^B \Big|_{\bar{\phi}_L} \langle \phi^2 \rangle_{(L)}^{(M)} = \left\langle N^A N_A \left(\bar{\phi}_L + \phi(\mathbf{x})_{(L)}^{(M)} \right) \right\rangle$$

3-point Function

$$\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\rangle = \Bigg[\left\langle N_{AB}N^A N^B \left(\bar{\phi}_L + \phi(\mathbf{x})^{(M)}_{(L)}\right)\right\rangle$$

$$+ \left(\tfrac{1}{2}N^A_{ABC}N^B N^C + N^A_{AB}N^{BC}N_C\right)\langle\phi^2\rangle^{(l)}_{(M)}\Bigg]\left(\tilde{G}_{12}\tilde{G}_{13}+\ldots\right)$$

$$+\tfrac{1}{2}N_{ABC}N^{AB}N^C\left(\left(\tilde{G}_{12}+\tilde{G}_{13}\right)\tilde{G}_{12}\tilde{G}_{13}+\ldots\right) + N_{AB}N^{BC}N^A_C\;\tilde{G}_{12}\tilde{G}_{13}\tilde{G}_{23}$$

$$+\underbrace{2N_{AB}N^A N^B\langle\phi^2\rangle^{(M)}_{(L)}\left(\tilde{G}_{12}+\tilde{G}_{13}+\tilde{G}_{23}\right)+\text{ const}}_{\text{Disconnected}}$$

Conjecture: For $\mathcal{A}(L)$ a generic "ΔN coefficient" of an N-point correlator defined with an IR cutoff L

- $\mathcal{A}_L \rightarrow \mathcal{A}_M = \left\langle\mathcal{A}\left(\bar{\phi}_L + \phi(\mathbf{x})^{(M)}_{(L)}\right)\right\rangle$
- Disconnected parts appear

For a generic ΔN coefficient $\mathcal{A}$ (One loop): $\quad L \to M \;\Rightarrow\; \mathcal{A} \to \left\langle \mathcal{A}\left(\bar{\phi}_L + \phi(\mathbf{x})_{(L)}^{(M)}\right) \right\rangle$

In practice this means a renormalization flow:

$$M\frac{d\mathcal{A}}{dM} = -\frac{1}{2}\,(\mathcal{A})_B^B\;\;\mathcal{P}_\phi(1/M)$$

Thus, to one loop:

- $$M\frac{d}{dM}\Big[\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle_M - \langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_3)\rangle_M\Big] = 0$$

- $$M\frac{d}{dM}\Big[\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\rangle_M - \frac{1}{2}\Big(\langle\zeta(\mathbf{x}_1)^2\zeta(\mathbf{x}_2)\rangle_M + \langle\zeta(\mathbf{x}_2)^2\zeta(\mathbf{x}_3)\rangle_M + \langle\zeta(\mathbf{x}_3)^2\zeta(\mathbf{x}_1)\rangle_M - \langle\zeta(\mathbf{x})^3\rangle_M\Big)\Big] = 0$$

<u>An Interpretation:</u> $L \to M \;\Rightarrow\; \mathcal{A} \to \left\langle \mathcal{A}\left(\bar\phi_L + \phi(\mathbf{x})^{(M)}_{(L)}\right) \right\rangle$

In a sense, the size of the box M represents simply the renormalization point, the redefinitions a renormalization prescription. As the cutoff is changed, the background theory is changed accordingly.

<u>Single field, monomial potential</u> $V(\phi) \propto \phi^n$: $N' = -\frac{1}{n}\frac{1}{M_{\rm P}^2}\bar\phi_\star$

Changing M simply shifts the background field value:

$$\bar\phi \to \bar\phi\sqrt{1 - \frac{\mathcal{P}}{\bar\phi^2}\ln H_0 M}$$

- One is always safe by using $1/H_0$ as an IR cutoff. However, the theory may not look the same as that defined on super-large scales.

  Since we know that our universe must have undergone ~ 60 inflationary e-folds, we can always assign a background field value $\bar{\phi}_\star$ corresponding to the time when our patch left the horizon. Changing the size of the box simply means changing $\bar{\phi}_\star$ slightly. (Presumably this holds for more complicated single field potentials)

- Can this procedure be applied for more scalar fields?

  Not clear. It is plausible however that with information on the iscurvature modes, one could find the appropriate shifts for all the fields. (This probably goes beyond ΔN).

For a free field : $\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle = (N')^2\, G_{12} + \frac{1}{2}(N'')^2 G_{12}^2 \quad \Rightarrow$

$$G_{12} = \bar{\phi}^2 \left(-1 + \sqrt{1 + (8\pi^2)^{-1}(M_{\rm P}/\bar{\phi})^4 \langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle}\,\right)$$

The difference $G_{12} - G_{13}$ is IR finite and assuming approximate scale invariance:

$$\sqrt{1 + (8\pi^2)^{-1}(M_{\rm P}/\bar{\phi})^4 \langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle} - \sqrt{1 + (8\pi^2)^{-1}(M_{\rm P}/\bar{\phi})^4 \langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_3)\rangle} = \frac{\mathcal{P}_\phi}{\bar{\phi}^2} \ln\left(\frac{r_{13}}{r_{12}}\right)$$

is a non-perturbative expression, manifestly independent of the cutoff.

A measurement of $\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\rangle$ and theoretical knowledge of $\mathcal{P}_\phi$ lead to an unambiguous and renormalization-point independent prediction for $\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_3)\rangle\ \forall\ \mathbf{x}_3$

Conclusions

- Stochastic inflation regulates IR fluctuations for slow roll. Only regions which have thermalized are interesting observationally.
- Predictions for the curvature perturbation on the largest possible thermalized scales can be related to observations performed in smaller regions by appropriate redefinitions of the coefficients in the ΔN expansion, up to the appearance of disconnected parts in the correlators. This procedure can be carried out to the case of 2-point and 3-point functions - it was conjectured that this is so for any n-point correlator.
- In a sense, the size of the box M represents simply the renormalization point, the redefinitions a renormalization prescription.
- It is therefore permissible to use our horizon as a cutoff in loop calculations.
- One can construct quantities which are manifestly independent of the cutoff.